On finite and elementary generation of $SL_2(R)$

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Abstract

Motivated by a question of A. Rapinchuk concerning general reductive groups, we are investigating the following question: Given a finitely generated integral domain R with field of fractions F, is there a *finitely generated subgroup* Γ of $SL_2(F)$ containing $SL_2(R)$? We shall show in this paper that the answer to this question is negative for any polynomial ring R of the form $R = R_0[s,t]$, where R_0 is a finitely generated integral domain with infinitely many (non–associate) prime elements. The proof applies Bass–Serre theory and reduces to analyzing which elements of $SL_2(R)$ can be generated by elementary matrices with entries in a given finitely generated R–subalgbra of F. Using Bass–Serre theory, we can also exhibit new classes of rings which do not have the GE_2 property introduced by P.M. Cohn.

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1 Introduction

The starting point of the present paper was the following problem raised by Andrei Rapinchuk (see [9], last paragraph):

Question 1.1 Given a finitely generated integral domain R with field of fractions F and a reductive F–group $\mathcal{G}$, does there exist a *finitely generated* subgroup of $\mathcal{G}(F)$ which contains $\mathcal{G}(R)$?

The background of this question is the following. In [8] Rapinchuk, Segev and Seitz prove the beautiful theorem that any finite quotient of the multiplicative group of a finite dimensional division algebra D is solvable. This leads them to the question whether any finite quotient of $\mathcal{G}(F)$ for a reductive group $\mathcal{G}$ over an infinite field F is solvable. Their result shows that this is true for $\mathcal{G} = GL_{1,D}$, and the obvious next candidate is $\mathcal{G} = SL_{1,D}$. However, the transition from $GL_{1,D}$ to $SL_{1,D}$ is non–trivial and involves the question whether a normal subgroup of finite index in $SL_{1,D}(F)$ contains a finite index subgroup which is normal in $GL_{1,D}(F)$. Rapinchuk could answer this question in the affirmative provided that Question 1.1 has a positive answer for $\mathcal{G} = SL_{1,D}$.

However, even without this background, Question 1.1 is interesting and challenging. It is certainly well–known that it has a positive answer for S-arithmetic groups $\mathcal{G}(R)$, which in "almost all" cases are finitely generated themselves (see Remark 4.4 below for more precise statements). But it was not clear whether one could expect a positive answer to Question 1.1 for arbitrary finitely generated integral domains R. A standard reduction in algebraic K-theory provides, modulo a (difficult) problem concerning the finite generation of $K_1(R)$ for regular R, some evidence that Rapinchuk's question admits a positive answer for $\mathcal{G}(R) = SL_n(R)$ in case n is "sufficiently large" (see Question 5.5 and Remark 5.6).

On the other hand, it seemed unlikely to me that Question 1.1 had a positive answer for $\mathcal{G} = SL_2$. The present paper is (mainly) about turning this vague idea into a rigorous proof for a reasonable class of rings R. This is the following theorem which will be proved in Section 4 (see Theorem 4.7).

Theorem 1.2 *Let R_0 be a finitely generated integral domain with infinitely many non–associate primes, $R = R_0[s,t]$ with field of fractions F and Γ a group with $SL_2(R) \leq \Gamma \leq SL_2(F)$. Then Γ is not finitely generated.*

The strategy is the following. One starts by making the elementary observation that a finitely generated group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F)$ exists if and only if $SL_2(R) \subseteq E_2(S)$ for some finitely generated R–subalgebra S of F (see Lemma 4.3; $E_2(S)$ denotes the subgroup of $SL_2(S)$ generated by elementary matrices). So for any given S, one wants to exhibit an element of $SL_2(R)$ which is not contained in $E_2(S)$. To this end, one provides F with an appropriate valuation, let $G = SL_2(F)$ act on the corresponding

(Bruhat–Tits) tree T, and considers the subgroup H_0 of $H = SL_2(S)$ generated by the stabilizers in H of the two vertices of a fundamental edge of T. By another elementary observation (Lemma 3.2), H_0 contains $E_2(S)$. Now Bass-Serre theory provides us with criteria to decide whether $H = H_0$ and with a method to construct a *concrete* element h of H not in H_0 if $H \neq H_0$ (see Lemmas 2.4 – 2.6). These general criteria now have to be applied in the given situation (which also requires a bit of commutative algebra), finally yielding matrices $h \in SL_2(R)$ with $h \notin E_2(S)$.

It turns out that this method is also effective in order to establish, under certain conditions, that $SL_2(R[1/\pi]) \neq E_2(R[1/\pi])$ for a (not necessarily finitely generated) integral domain R with prime element π. More precisely, we obtain the following theorem which will be proved, among other results, in Section 3 (see Corollary 3.5).

Theorem 1.3 *Let R be an integral domain and π a prime element of R satisfying $\bigcap_{n\geq 0} \pi^n R$ $= \{0\}$. If $R/\pi R$ is not a Bezout domain or if the canonical homomorphism $SL_2(R) \to SL_2(R/\pi R)$ is not surjective, then $SL_2(R[1/\pi]) \neq E_2(R[1/\pi])$.*

This generalizes results about Laurent polynomial rings proved in [1] and [5]. Here (that is in Theorem 3.4) we again investigate the question whether $H = H_0$, with $H = SL_2(R[1/\pi])$ acting on the Bruhat–Tits tree associated to $SL_2(F)$ and π, and $E_2(R[1/\pi]) \leq H_0$. In fact we can derive a necessary and sufficient condition for $H = H_0$ in this situation.

The paper is organized as follows. In Section 2 we prove some lemmas about subgroups of amalgams, using the action of these groups on the associated trees. This provides us with the above mentioned criteria concerning $H = H_0$, $H \neq H_0$. We first apply these criteria in Section 3 in order to deduce a necessary and sufficient condition for $SL_2(R[1/\pi]) = H_0$, where H_0 is the subgroup generated by $SL_2(R)$ and its conjugate by the diagonal matrix with entries $1/\pi$ and 1. In Section 4 we deduce the negative answer to Rapinchuk's problem for the groups $SL_2(R_0[s,t])$ in the way indicated above. We conclude this paper by listing some further questions and conjectures in Section 5.

Acknowledgements: I am grateful to my colleagues Andrei Rapinchuk and Nicholas Kuhn for stimulating discussions on this subject. Andrei's problem for reductive groups motivated the whole paper, and Section 3 emerged from an interesting question which Nick asked me during one of my talks.

2 About subgroups of amalgams

In this section, we consider the following set–up. The group G is the free product with amalgamation of its two subgroups A and B, amalgamated along their intersection $U = A \cap B$, H is an arbitrary subgroup of G and H_0 is the subgroup of H generated by $(A \cap H) \cup (B \cap H)$. We use the notations

$$G = A *_U B \quad , \quad H \leq G \quad \text{and} \quad H_0 = \langle (A \cap H) \cup (B \cap H) \rangle$$

We are interested in the following

Question 2.1 When is $H = H_0$, and when is $H \neq H_0$?

Structure theorems for subgroups of amalgams have been known in combinatorial group theory for a long time, see for instance [7]. However, Question 2.1 is attacked in a less technical and more transparent way by using group actions on trees. If X is a tree, we denote by VX its set of vertices and by EX its set of edges. Here an edge is always understood as a geometric edge, i.e. it is identified with a subset of cardinality 2 of VX. If a group C acts (on the left) on X, then we denote this action with a dot and set $C_\alpha := \{c \in C \mid c.\alpha = \alpha\}$ for any vertex or any edge α of X. Let us first recall one of the basic results about amalgams (cf. [10, Chapter I, Section 4.1]).

Fact 2.2 *G acts without inversion on a (suitable) tree T with an edge $e = \{x, y\}$ as fundamental domain (i.e. $G.e = ET$, $G.x \cup G.y = VT$ and $G.x \neq G.y$) such that $G_x = A$, $G_y = B$ and $G_e = U$.*
Conversely, if a group G' acts without inversion on a tree T' with an edge $e' = \{x', y'\} \in ET'$ as fundamental domain, then $G' = G_{x'} *_{G'_{e'}} G'_{y'}$.

The second statement in Fact 2.2 has a well-known generalization (cf. [10, Chapter I, Section 4.5]).

Fact 2.3 *If a group G' acts on a tree T' with a subtree T_1 of T as fundamental domain, and if T_1 is considered as a tree of groups with respect to the system $\mathcal{G} = \big((G'_v)_{v \in VT_1}, (G'_f)_{f \in ET_1}\big)$, then G' is canonically isomorphic to the direct limit (which is an "amalgam along T_1") $\lim(\mathcal{G}, T_1)$ of this tree of groups.*

We now fix T and e as in Fact 2.2. In the following sequence of three lemmas dealing with Question 2.1, the first one is similar to some well–known results. However, for the convenience of the reader I shall give a short proof also in this case.

Lemma 2.4 *Denote by X the subforest of T with edge set $H.e$ and vertex set $H.x \cup H.y$.*

(i) If X is connected, then $H = (A \cap H) *_{(U \cap H)} (B \cap H)$, and in particular $H = H_0$.

(ii) If $A = (A \cap H)U$ and $B = (B \cap H)U$, then $X = T$, and hence $H = H_0$ by (i).

Proof. Observe that H acts on T with stabilizers $H_x = G_x \cap H = A \cap H$, $H_y = B \cap H$ and $H_e = U \cap H$. Now (i) immediately follows from the second part of Fact 2.2 if we set $G' = H$ and $T' = X$, which is a tree by assumption.

In order to prove (ii), we have to show $H.e = ET$. Given $e' \in ET$, we consider the geodesic $\gamma = (z = z_0, z_1, \ldots, z_n)$ in T with $z_i \in VT$ for all i, $\{z_i, z_{i+1}\} \in ET$ for all $i \leq n-1$, $z_i \neq z_{i+2}$ for all $i \leq n-2$, $\{z_0, z_1\} = e$ and $\{z_{n-1}, z_n\} = e'$. We show $e' \in H.e$ by induction on n. We may assume $z_1 = x$ (the case $z_1 = y$ is similar) and $n \geq 2$. By assumption, $H_x G_e = G_x$. Now G_x acts transitively on the set of edges containing x because e is a fundamental domain for the action of G on T. Hence there exists an $h \in H_x$ such that $h.\{z_1, z_2\} = e$. Applying the induction hypothesis to the geodesic $(x = h.z_1, y = h.z_2, \ldots, h.z_n)$, we obtain $h.e' \in H.e$, which immediately implies $e' \in H.e$. □

If one of the two assumptions in Lemma 2.4(ii) is not satisfied, then H is "often" different from H_0, as the following result shows.

Lemma 2.5 *Assume that the following two conditions are satisfied.*

(1) There exists $a \in A$ with $a \notin (A \cap H)U$.

(2) There exists $b \in B$ with $b \notin U$ and $aba^{-1} \in H$.

Then $H \neq H_0$.

Proof. Note first that the edges e and $a.e$ are in different H–orbits since $a.e \in H.e$ implies $a \in HU \cap A = (A \cap H)U$, contradicting (1). Because the element $h := aba^{-1} \in aBa^{-1} = G_{a.y}$ is also in H but not in $aUa^{-1} = G_{a.e}$ by assumption (2), we have $h \in H_{a.y}$ and $h \notin H_{a.e}$. We now distinguish two cases.

First Case: $a \in HB$**.** Here $a.y$ is contained in $H.y$. So the images of y and $a.y$ are equal in the quotient graph $H \setminus T$. Since $a \in G_x$, we also have $x = a.x$. But as observed above, the edges e and $a.e$ do not have the same image in $H \setminus T$. Therefore $H \setminus T$ contains a circuit of length 2. It now follows from [10, Chapter I, Section 5.4, Corollary 1 of Theorem 13] that $H \neq \langle \bigcup_{v \in VT} H_v \rangle$, hence in particular $H \neq H_0$.

Second Case: $a \notin HB$**.** So the H–orbits of $a.y$ and y are different. Hence the subtree T_0 of T with $VT_0 = \{y, x, a.y\}$ and $ET_0 = \{e, a.e\}$ is mapped injectively into $H \setminus T$ by the canonical projection $T \to H \setminus T$. Now consider the subgroup $H' := \langle H_y \cup H_x \cup H_{a.y} \rangle$ of H which acts on $T' := H'.T_0$. We show that T' is connected and hence a subtree of T. For any integer $n \geq 1$, we set $T_n := \bigcup h_1 \dots h_n.T_0$, where $h_1 \dots h_n$ runs over all products with factors $h_i \in (H_y \cup H_x \cup H_{a.y})$ for all $1 \leq i \leq n$. For any such product, the intersection $h_1 \dots h_n.T_0 \cap T_{n-1}$ is obviously nonempty. So by induction, T_n is connected for all n. Hence also $T' = \bigcup_{n \geq 0} T_n$ is connected. By construction and since T_0 embeds into $H \setminus T$, hence also into $H' \setminus T'$, T_0 is a fundamental domain for the action of H' on T'. So by Fact 2.3, H' is the direct limit of the tree of groups associated with T_0 and $\big((H_y, H_x, H_{a.y}),\, (H_e, H_{a.e})\big)$, showing

$$H' = (H_y *_{H_e} H_x) *_{H_{a.e}} H_{a.y} = H_0 *_{H_{a.e}} H_{a.y}$$

As observed above, $h \in H_{a.y}$ and $h \notin H_{a.e}$. The normal form for amalgams (cf. [10, Chapter I, Section 1.2]) now yields $h \notin H_0$. Therefore $H \neq H_0$. □

The proof of Lemma 2.5 yields additional information which is worth mentioning.

Lemma 2.6 *Assume that the following two conditions are satisfied.*

(1) There exists $a \in A$ with $a \notin HB$.

(2) There exists $b \in B$ with $b \notin U$ and $aba^{-1} \in H$.

Then $h := aba^{-1}$ is not an element of H_0.

Proof. $a \notin HB$ obviously implies $a \notin (A \cap H)U$. So the assumptions of Lemma 2.5 are satisfied, and additionally we are in Case 2 of its proof. As demonstrated there, this implies $h \notin H_0$. □.

In the application which we shall discuss in the last section it will become important that Lemma 2.6 provides us with a method that produces *concrete* elements in H which are not contained in H_0. It turns out that Condition (2) can be trivially satisfied in those situations where we are going to apply Lemma 2.5 and Lemma 2.6. However, some work will be necessary in order to verify Condition (1).

3 Non–elementary generation of $SL_2(R[1/\pi])$

Let R be an integral domain with field of fractions F and $\pi \in R$ a prime element. In this section we shall deduce some necessary conditions for $SL_2(R[1/\pi])$ to be generated by elementary matrices. We start with an easy exercise in commutative algebra which we shall need later on.

Lemma 3.1 *Let $u, v, x, y \in R$ with $yu \neq 0$ and $ux = vy$. Then (u, v) is a principal ideal of R if and only (x, y) is.*

Proof. By symmetry we may assume that $(u, v) = (d)$ with $d \in R$. Then $d \neq 0$ (since $u \neq 0$), and $u_1 := u/d \in R$ as well as $v_1 := v/d \in R$. Also there exist $r, s \in R$ with $ru + sv = d$, hence $ru_1 + sv_1 = 1$. We claim that $(x, y) = (sx + ry)$. So we have to show $x, y \in (sx + ry)$. Recall that $ux = vy$, hence $u_1 x = v_1 y$. So we obtain $x = (ru_1 + sv_1)x = rv_1 y + sv_1 x = v_1(sx + ry)$ and $y = (ru_1 + sv_1)y = ru_1 y + su_1 x = u_1(sx + ry)$. This proves the claim. $\square$

With respect to elementary matrices, we shall use the following notations. We set

$$E_{12}(r) := \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad E_{21}(r) := \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad \text{for any } r \in R$$

and then $E_{12}(R) := \{E_{12}(r) \mid r \in R\}$, $E_{21}(R) := \{E_{21}(r) \mid r \in R\}$ as well as $E_2(R) := \langle E_{12}(R) \cup E_{21}(R) \rangle$. For any two $\alpha, \beta \in F^*$, we define $D(\alpha, \beta) := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in GL_2(F)$.

The following observation concerning the ring $R[1/r]$ is elementary but useful. It was already successfully applied in [1].

Lemma 3.2 *For any $r \in R$ with $r \neq 0$ we obtain the following:*

(i) $E_2(R[1/r]) = \langle E_2(r) \cup \{E_{12}(1/r)\} \rangle$

(ii) $E_2(R[1/r]) \leq \langle SL_2(R) \cup D(1/r, 1) SL_2(R) D(r, 1) \rangle \leq SL_2(R[1/r])$

Proof. (i) follows from the well–known identity

$$(E_{12}(1/r)E_{21}(-r)E_{12}(1/r))(E_{12}(-1)E_{21}(1)E_{12}(-1)) = D(1/r, r)$$

together with $D(1/r,r)^n E_{12}(R) D(1/r,r)^{-n} = E_{12}(r^{-2n}R)$ and $D(1/r,r)^{-n} E_{21}(R) D(1/r,r)^n = E_{21}(r^{-2n}R)$ for all integers n. The first inclusion in (ii) follows immediately from (i), and the second is obvious. $\square$

So $SL_2(R[1/r])$ can only be generated by elementary matrices if it is also generated by $SL_2(R) \cup D(1/r,1) SL_2(R) D(r,1)$. It is the question whether $SL_2(R[1/r]) = \langle SL_2(R) \cup D(1/r,1) SL_2(R) D(r,1) \rangle$ or not to which we can apply our results from Section 2. More precisely, we shall do this in case $r = \pi$ is a prime element in R in order to have a nice action of $SL_2(R[1/r])$ on a suitable Bruhat–Tits tree (see Fact 3.3 below). Before we can introduce the latter, we need the following assumption, which is obviously satisfied for all noetherian, and hence also for all finitely generated rings.

Assumption (A): The prime element $\pi \in R$ satisfies $\bigcap_{n \geq 0} \pi^n R = \{0\}$.

Now let (A) be satisfied for a fixed prime $\pi \in R$. We define a π–adic valuation $v = v_\pi$ on F in the usual way. For any $r \in R \setminus \{0\}$, we set $v(r) := \max\{n \geq 0 \mid r \in (\pi^n)\}$, which exists in view of (A). We further define $v(x/y) := v(x) - v(y)$ for $x, y \in R \setminus \{0\}$ and $v(0) := \infty$. It is immediately verified that v is thus a *discrete valuation* on F. We denote by $\mathcal{O}$ the associated discrete valuation ring $\mathcal{O} = \{\alpha \in F \mid v(\alpha) \geq 0\}$, by $\mathcal{P} = \pi\mathcal{O}$ its maximal ideal and by $\mathcal{O}^* = \mathcal{O} \setminus \mathcal{P}$ its group of units. Our reference for the following statements is again Serre's book; cf. [10, Chapter II, Section 1].

Fact 3.3 *Given F together with the disrete valuation v, one can construct a tree T (which is also the Bruhat-Tits building of $SL_2(F)$ with respect to v) on which $G = SL_2(F)$ acts without inversion and with an edge as fundamental domain. This edge e can be chosen such that the stabilizers of its two vertices are $A = SL_2(\mathcal{O})$ and $B = D(1/\pi,1) SL_2(\mathcal{O}) D(\pi,1)$, respectively, and $G_e = U = A \cap B = SL_2 \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} \end{pmatrix}$. By Fact 2.2 this implies $G = A *_U B$.*

We now want to apply the results of Section 2 to this situation. Recall that a commutative ring is called Bezout if each of its finitely generated ideals is a principal ideal.

Theorem 3.4 *If Assumption (A) is satisfied, then the following two statements are equivalent:*

(1) $SL_2(R[1/\pi]) = \langle SL_2(R) \cup D(1/\pi,1) SL_2(R) D(\pi,1) \rangle$

(2) *$R/\pi R$ is a Bezout domain and the canonical homomorphism $SL_2(R) \to SL_2(R/\pi R)$ is surjective*

Proof. We consider the subgroup $H = SL_2(R[1/\pi])$ of $G = SL_2(F)$. With A, B and U as in Fact 3.3, we obtain $A \cap H = SL_2(R)$, $B \cap H = D(1/\pi, 1)(A \cap H)D(\pi, 1) = D(1/\pi, 1)SL_2(R)D(\pi, 1)$ and $U \cap H = SL_2\begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}$. Setting $H_0 = \langle (A \cap H) \cup (B \cap H) \rangle = \langle SL_2(R) \cup D(1/\pi, 1)SL_2(R)D(\pi, 1) \rangle$, we have to answer Question 2.1 in this situation.

The implication "(2) ⇒ (1)". We shall show that $A = (A \cap H)U$ and $B = (B \cap H)U$ if (2) is satisfied. Then (1) will follow from Lemma 2.4. So, firstly, given any $a = \begin{pmatrix} \alpha & * \\ \beta & * \end{pmatrix} \in A = SL_2(\mathcal{O})$, we have to find an $h = \begin{pmatrix} * & * \\ r & s \end{pmatrix} \in A \cap H = SL_2(R)$ such that $ha \in U$, i.e. such that $r\alpha + s\beta \in \mathcal{P}$. Since $\alpha, \beta \in \mathcal{O}$, there exists $z \in R \cap \mathcal{O}^*$ such that $p := z\alpha, q := z\beta$ are both elements of R. Denote by $\overline{p}$ and $\overline{q}$ the respective images in $R/\pi R$. Because this ring is Bezout by assumption, $(\overline{p}, \overline{q})$ is a prinicipal ideal, i.e. $(\overline{p}, \overline{q}) = (\delta)$ for some $\delta \in R/\pi R$. Note that $\delta \neq \overline{0}$ since $\mathcal{O} = \alpha\mathcal{O} + \beta\mathcal{O} = p\mathcal{O} + q\mathcal{O}$. Set $\lambda := \overline{q}/\delta$, $\mu := -\overline{p}/\delta \in R/\pi R$. Then $(\lambda, \mu) = (\overline{1})$, and we can find a matrix $\begin{pmatrix} * & * \\ \lambda & \mu \end{pmatrix} \in SL_2(R/\pi R)$. By assumption, this matrix has a preimage $\begin{pmatrix} * & * \\ r & s \end{pmatrix}$ in $SL_2(R)$; we call this preimage h. Now by construction, $\overline{rp} + \overline{sq} = \lambda\overline{p} + \mu\overline{q} = (\overline{qp} - \overline{pq})/\delta = \overline{0}$. Therefore, $rp + sq \in \pi R$, and hence also $r\alpha + s\beta \in \mathcal{P}$, because $z \in \mathcal{O}^*$. This proves that $ha \in U$.

The equation $B = (B \cap H)U$ is equivalent to $A = (A \cap H)D(\pi, 1)UD(1/\pi, 1)$. Now this equation can be proved completely similar as the equation $A = (A \cap H)U$ above. We only have to produce a $(1, 2)$–entry in $\mathcal{P}$ for the product ha instead of a $(2, 1)$–entry in $\mathcal{P}$.

The implication "(1) ⇒ (2)". Now we assume that $R/\pi R$ is not Bezout or that the canonical homorphism $\phi : SL_2(R) \to SL_2(R/\pi R)$ is not surjective. In the first case, we choose elements $x, y \in R$ such that the ideal $(\overline{x}, \overline{y})$ of $R/\pi R$ is not principal. In the second case, we choose $x, y \in R$ such that there exists a matrix $k \in SL_2(R/\pi R)$ of the form $k = \begin{pmatrix} \overline{y} & * \\ \overline{x} & * \end{pmatrix}$ with $k \notin \operatorname{im}\phi$. Note that $x, y \in \mathcal{O}^*$ in both cases. This is obvious if $(\overline{x}, \overline{y})$ is not principal, and in the second case it follows from $E_2(R/\pi R) \subseteq \operatorname{im}\phi$ and the easy observation that $k \in E_2(R/\pi R)$ if one of the entries of k is equal to $\overline{0}$.

Having chosen x and y, we now define $a := \begin{pmatrix} 1 & 0 \\ x/y & 1 \end{pmatrix} \in A$. Suppose $a \in (A \cap H)U$. Then, as above, there is an $h = \begin{pmatrix} * & * \\ r & s \end{pmatrix} \in SL_2(R)$ such that $ha \in U$. This implies

$r+sx/y \in \mathcal{P}$, hence $ry+sx \in \mathcal{P} \cap R = \pi R$ and thus $\overline{ry}+\overline{sx} = \overline{0}$ in $R/\pi R$. However, this contradicts our choice of x and y in both cases. Firstly, $\overline{ry} = -\overline{sx}$ and Lemma 3.1 imply that $(\overline{x}, \overline{y})$ is principal because $(\overline{r}, -\overline{s}) = (\overline{1})$ is principal. Secondly, $\phi(h)k$ is of the form $\phi(h)k = \begin{pmatrix} * & * \\ \overline{0} & * \end{pmatrix}$. Hence $\phi(h)k \in E_2(R/\pi R) \subseteq \operatorname{im}\phi$, implying $k \in \operatorname{im}\phi$.

So in both cases, $a \in A$ and $a \notin (A \cap H)U$. We now set $b := \begin{pmatrix} 1 & y^2/\pi \\ 0 & 1 \end{pmatrix} \in B$. We have $b \notin U$ since $y \in \mathcal{O}^*$. And we have that $aba^{-1} = \begin{pmatrix} 1-xy/\pi & y^2/\pi \\ -x^2/\pi & 1+xy/\pi \end{pmatrix} \in SL_2(R[1/\pi] = H$. Therefore, $H \neq H_0$ by Lemma 2.5. □

From my point of view, the most interesting consequence of Theorem 3.4 (and of the elementary Lemma 3.2) is the following, which was stated as Theorem 1.3 in the Introduction.

Corollary 3.5 *If R is an integral domain and $\pi \in R$ a prime element satisfying Assumption (A), then $SL_2(R[1/\pi]) \neq E_2(R[1/\pi])$ whenever $R/\pi R$ is not Bezout or the canonical homomorphism $SL_2(R) \to SL_2(R/\pi R)$ is not surjective.*

□

Remark 3.6 If R is noetherian, then Assumption (A) is automatically satisfied and hence superfluous in the statement of Theorem 3.4 as well as in Corollary 3.5. Furthermore, "Bezout" can be equivalently replaced with "principal ideal domain" in this case.

A special case of Corollary 3.5 is obtained if $R = R_0[t]$ is the polynomial ring in one variable over an integral domain R_0 and $\pi = t$, in which case Assumption (A) is clearly satisfied and the canonical homorphism $SL_2(R) \to SL_2(R/\pi R) = SL_2(R_0)$ always surjective. So we recover the following result about Laurent polynomial rings which was partly deduced by Bachmuth-Mochizuki in [1] and first proved in the generality we state it here by H. Chu (cf. [5]).

Corollary 3.7 *If R_0 is an integral domain which is not Bezout, then $SL_2(R_0[t,t^{-1}]) \neq E_2(R_0[t,t^{-1}])$.*

□

One remarkable feature about Corollary 3.5 is that $R[1/\pi]$ cannot be a GE_2–ring in the sense of Cohn (cf. [6]), no matter how "nice" R is, if the quotient ring $R/\pi R$ does not have the stated properties. So, in particular, the GE_2–property is not preserved by the process of "localization", by which we mean the transition from a ring to one of its rings of fractions. This is well demonstrated by the following example and answers a respective question of Nick Kuhn.

Example 3.8 Let R be a (noetherian) regular local ring of Krull dimension ≥ 3. (Take for instance the localization $R = S_{\mathcal{M}}$ of the polynomial ring $S = K[t_1, t_2, t_3]$ over a field K at the maximal ideal $\mathcal{M} = (t_1, t_2, t_3)$.) Because R is local, $SL_2(R) = E_2(R)$. However, for any prime element $\pi \in R$ (and R has a lot of prime elements since it is a unique factorization domain), $R/\pi R$ has Krull dimension ≥ 2. Hence $R/\pi R$ is not a principal ideal domain, and so $SL_2(R[1/\pi]) \neq E_2(R[1/\pi])$ by Corollary 3.5.

So far we have been discussing consequences of Theorem 3.4 concerning the elementary generation of $SL_2(R[1/\pi])$. Let us finish this section by mentioning two cases where Condition (2) is obviously satisfied.

Corollary 3.9 *Let R be a ring which is either a Dedekind domain or of the form $R = R_0[t]$ with a Bezout domain R_0. Let π be an arbitrary prime element of R in the first case and $\pi = t$ in the second case. Then $SL_2(R[1/\pi]) = \langle SL_2(R) \cup D(1/\pi, 1) SL_2(R) D(\pi, 1)\rangle$, and moreover*

$$SL_2(R[1/\pi]) = SL_2(R) *_U SL_2 \begin{pmatrix} R & \pi^{-1}R \\ \pi R & R \end{pmatrix} \text{ with } U = SL_2 \begin{pmatrix} R & R \\ \pi R & R \end{pmatrix}$$

Proof. $R/\pi R$ is a field in the first, and $R/\pi R = R_0$ in the second case. Hence Condition (2) of Theorem 3.4 is satisfied, yielding the first claim of this corollary. However, the proof of Theorem 3.4 in fact shows that the assumptions of Lemma 2.4(ii) are satisfied. Therefore, this lemma implies the second claim about the amalgam presentation of $SL_2(R[1/\pi])$. □

For Dedekind rings, a different proof of Corollary 3.9 is given in [10, Chapter II, Section 1.4]. (It is stated there only for $R = \mathbb{Z}$ but could be generalized.) The result about Laurent polynomial rings is essentially Theorem 2 in [1].

Remark 3.10 Whenever Condition (2) of Theorem 3.4 is satisfied, the proof of the implication "(2) $\Rightarrow$ (1)" together with Lemma 2.4 yields the same amalgam presentation of $SL_2(R[1/\pi])$ as stated in Corollary 3.9.

Remark 3.11 Corollary 3.9 is not of much help in order to decide the question whether Laurent polynomial rings in one variable over principal ideal domains are elementary generated. To the best of my knowledge, it is still an open problem whether the groups $SL_2(\mathbb{Z}[t,t^{-1}])$ and $SL_2(\mathbb{F}_q[t_1,t_1^{-1};t_2,t_2^{-1}])$ are generated by elementary matrices. Even the weaker question whether they are finitely generated still seems to be open. (It is easily seen that $E_2(\mathbb{Z}[t,t^{-1}])$ and $E_2(\mathbb{F}_q[t_1,t_1^{-1};t_2,t_2^{-1}])$ are finitely generated; see the proof of Lemma 4.2(ii) below.)

4 Non–finite generation of groups between $SL_2(R)$ and $SL_2(F)$

We now turn to the problem which motivated this paper. In this section, R will always denote a finitely generated integral domain, i.e. an integral domain which is finitely generated as a $\mathbb{Z}$–algebra. So R can be obtained by adjoining finitely many elements to its prime ring P ($P = \mathbb{Z}$ or $P = \mathbb{F}_p$), that is $R = P[x_1,\ldots,x_n]$ with elements $x_1,\ldots,x_n \in R$. We denote by F the field of fractions of R, so $F = Q(x_1,\ldots,x_n)$ with $Q = \mathbb{Q}$ or $Q = \mathbb{F}_p$. Let us recall the question we want to answer:

Question 4.1 Does there exist a *finitely generated* group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F)$?

There is an intimate connection between finite and elementary generation, as the following easy lemma shows.

Lemma 4.2 *The following holds:*

(i) Any finitely generated subgroup of $SL_2(F)$ is contained in $E_2(S)$ for some finitely generated subring S of F.

(ii) Any finitely generated subring S of F is included in some finitely generated S–subalgebra $S' \subseteq F$ for which $E_2(S')$ is a finitely generated group.

Proof. We again denote by $P = \mathbb{Z}$, respectively $P = \mathbb{F}_p$, the prime subring of F.

(i) Assume that $\Gamma = \langle \gamma_1, \dots, \gamma_l \rangle$ is a finitely generated subgroup of $SL_2(F)$. Recall that $SL_2(F) = E_2(F)$. For each $1 \leq i \leq l$, we fix a representation of γ_i as a product $\gamma_i = e_{i1} \dots e_{ik_i}$ of elementary matrices e_{ij} ($1 \leq j \leq k_i$) in $SL_2(F)$. Let M_i be the finite subset of F consisting of all entries of all the e_{ij}. Set $M := \bigcup_{i \leq l} M_i$ and $S := P[M]$. Then, by construction, S is a finitely generated subring of F and $\Gamma = \langle \gamma_1, \dots, \gamma_l \rangle \leq E_2(S)$.

(ii) Now suppose that $S = P[y_1, \dots, y_m]$ with $y_i \in F^*$ for all $1 \leq i \leq m$. We set $S' := S[y_1^{-1}, \dots, y_m^{-1}]$. An easy calculation (using conjugation of elementary matrices by diagonal matrices) shows that $E_2(S')$ is generated by the diagonal matrices $D(y_i, y_i^{-1})$, $1 \leq i \leq m$, together with the elementary matrices $E_{12}(z)$, $E_{21}(z)$, where z runs over all products of the form $z = y_{i_1} \dots y_{i_k}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq m$, including the empty product ($k = 0$) which is 1 by definition. In particular, $E_2(S')$ is finitely generated. $\square$

This admits a reformulation of Question 4.1 in terms of elementary generation.

Lemma 4.3 *With R and F as above, the following two statements are equivalent:*

(1) There exists a finitely generated group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F)$.

(2) There exists a finitely generated R–subalgebra $S \subseteq F$ such that $SL_2(R) \leq E_2(S)$.

Proof. If (1) is satisfied, then Γ and hence $SL_2(R)$ is contained in $E_2(S)$ for a finitely generated subring S of F by Lemma 4.2(i), and S has to contain R. If (2) is satisfied, then $SL_2(R)$ is contained in the finitely generated group $E_2(S')$ with S' chosen as in Lemm 4.2(ii). $\square$

Example 4.4 It is a classic result that $SL_n(\mathbb{Z}) = E_n(\mathbb{Z})$ is finitely generated for all positive integers n. It is also a well-known result due to Nagao that $SL_2(\mathbb{F}_q[t]) = E_2(\mathbb{F}_q[t])$ is not finitely generated. However, $SL_2(\mathbb{F}_q[t, t^{-1}])$ is of course finitely generated. More generally, for any S–arithmetic ring (also called a "Hasse domain" in the literature) $R = \mathcal{O}_S$, the group $SL_2(\mathcal{O}_S)$ is finitely generated whenever the characteristic of R is 0 or the set S of places has cardinality at least 2. This follows from general results about S–arithmetic groups due to Borel – Harish–Chandra in characteristic 0 and to Behr in characteristic $p > 0$ (see [4] and [3]). So Question 4.1 and, more generally, Question 1.1 have a positive answer for S–arithmetic rings.

In view of the last remark, we are now going to consider rings R with Krull dimension > 1. For (Laurent) polynomial rings in one variable over $\mathbb{Z}$ or in two variables over a finite field $\mathbb{F}_q$, Question 4.1 would involve the long standing problem mentioned in Remark 3.11, which we are not going to discuss in this paper. So it is natural to consider (Laurent) polynomial rings in at least two variables over infinite base rings in order to prove a negative answer to Question 4.1 for a reasonable class of rings. Let us fix some further notation:

Let R_0 be a finitely generated infinite integral domain with field of fractions F_0. F will be a transcendental extension of F_0 of transcendence degree 2, $F = F_0(s,t)$, and we start by considering $R = R_0[s,t,s^{-1},t^{-1}]$. (Lemma 4.2(ii) indicates that one should invert the variables in order to avoid trivial obstacles to finite generation. However, we shall later see that we can also replace this Laurent polynomial ring with a polynomial ring.) In view of Lemma 4.3, we want to show that $SL_2(R)$ is not contained in $E_2(S)$ for any given finitely generated R–subalgebra $S \subseteq F$. The idea is to write S as $\tilde{S}[t^{-1}]$ for a suitable subring $\tilde{S}$ of S and to apply a similar method as in Section 3 (see Corollary 3.5). However, it is not enough to just show $SL_2(S) \neq E_2(S)$ here. We need to be able to exhibit *concrete* elements in $SL_2(S)$ which are not in $E_2(S)$, and we must be able to choose these elements already in $SL_2(R)$. So we are going to apply Lemma 2.6 rather than Lemma 2.5 in the following. The main technical step in the proof is to verify Condition (1) of Lemma 2.6 in a suitable situation. This will be done in the framework of the following proposition.

Proposition 4.5 *Let $f = f(s,t) \in R_0[s,t]$ be a polynomial not divisible by t. Write $f = f_0(s) + f_1(s)t + \ldots f_d(s)t^d$ with polynomials $f_i \in R_0[s]$ for all $0 \le i \le d$. Set $S := R_0[s,t,s^{-1},t^{-1},f_0^{-1},f^{-1}]$ and $g := 1 - sf_0 \in R_0[s]$. Let p be any prime element of R_0 which does not divide f_0 in $R_0[s]$. Then we obtain*

$$\begin{pmatrix} 1 - pgt^{-1} & p^2t^{-1} \\ -g^2t^{-1} & 1 + pgt^{-1} \end{pmatrix} \notin E_2(S)$$

Proof. We first note that $f_0 \neq 0$ since t does not divide f. We put $\tilde{S} := R_0[s,t,s^{-1},f_0^{-1},f^{-1}]$ so that $S = \tilde{S}[t^{-1}]$. Note that t is a prime element of $\tilde{S}$ since it is a prime element of $R_0[s,t]$ which does not divide sf_0f. Replacing R with $\tilde{S}$ and π with t, we can now proceed as in Section 3. We introduce the t–adic valuation $v = v_t$ on F with associated discrete valuation ring $\mathcal{O}$, maximal ideal $\mathcal{P}$ and group of units $\mathcal{O}^*$. We have the same subgroups A, B, U of $G = SL_2(F)$ as introduced in Fact 3.3. Our $R[1/\pi]$ is S here, and hence we put $H = SL_2(S)$. Then $A \cap H = SL_2(\tilde{S})$, $B \cap H = D(t^{-1},1)SL_2(\tilde{S})D(t,1)$ and

$U \cap H = SL_2\begin{pmatrix} \tilde{S} & \tilde{S} \\ t\tilde{S} & \tilde{S} \end{pmatrix}$. We again set $H_0 = \langle (A \cap H) \cup (B \cap H) \rangle$ and recall that H_0 contains $E_2(\tilde{S}[t^{-1}]) = E_2(S)$ by Lemma 3.2. g and p are nonzero elements of $R_0[s]$ and hence also elements of $\mathcal{O}^*$. So we can consider the matrix $a = \begin{pmatrix} 1 & 0 \\ g/p & 1 \end{pmatrix} \in A = SL_2(\mathcal{O})$.

Claim: $a \notin HB$**.** We introduce another ring, namely $Z := R_0[s, s^{-1}, f_0^{-1}]$. Note that $Z \setminus \{0\} \subseteq \mathcal{O}^*$ since the elements of Z do not involve t. Hence the canonical homorphism $\phi : \tilde{S} \to \tilde{S}/t\tilde{S}$ restricted to Z is injective. $\phi\mid_Z$ is also surjective since $f \equiv f_0 \mod t$, implying $\phi(f) = \phi(f_0)$ and $\phi(f^{-1}) = \phi(f_0^{-1})$. (This was the reason for including f_0^{-1} in S.) We can thus decompose the additive group of $\tilde{S}$ as follows:

$(*)\quad \tilde{S} = Z \oplus t\tilde{S} = Z \oplus tZ \oplus t^2\tilde{S}$

We now assume by way of contradiction that $a \in HB$. Then there is a matrix $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in H = SL_2(S)$ such that $ha \in B$. Hence we have

$$\begin{pmatrix} \alpha + \beta g/p & \beta \\ \gamma + \delta g/p & \delta \end{pmatrix} \in SL_2\begin{pmatrix} \mathcal{O} & t^{-1}\mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix}$$

This implies $\delta \in \mathcal{O} \cap S = \tilde{S}$, $\beta \in t^{-1}\mathcal{O} \cap S = t^{-1}(\mathcal{O} \cap S) = t^{-1}\tilde{S}$, $\gamma \in \mathcal{O} \cap S = \tilde{S}$ (recall that $g/p \in \mathcal{O}^*$) and $\alpha \in t^{-1}\mathcal{O} \cap S = t^{-1}\tilde{S}$. Using the decomposition $(*)$, one therefore finds elements $a_{-1}, a_0, b_{-1}, b_0, c_0, c_1, d_0, d_1 \in Z$, $\alpha', \beta' \in t\tilde{S}$ and $\gamma', \delta' \in t^2\tilde{S}$ such that $\alpha = a_{-1}t^{-1} + a_0 + \alpha'$, $\beta = b_{-1}t^{-1} + b_0 + \beta'$, $\gamma = c_0 + c_1 t + \gamma'$ and $\delta = d_0 + d_1 t + \delta'$. But we still have the conditions $\alpha + \beta g/p \in \mathcal{O}$ and $\gamma + \delta g/p \in t\mathcal{O}$, which yield (together with $p, g \in Z$):

(1) $pa_{-1} + gb_{-1} = 0$

(2) $pc_0 + gd_0 = 0$

We also have the condition that $\det(h) = \alpha\delta - \beta\gamma = 1$ which leads to the equations $a_{-1}d_0 - b_{-1}c_0 = 0$ (which we do not need) and

(3) $a_{-1}d_1 + a_0 d_0 - b_{-1}c_1 - b_0 c_0 = 1$

(which we do need). Since p is prime in R_0, it is also prime in $R_0[s]$, and since p does not divide f_0 in $R_0[s]$ by assumption (and certainly not s), p is also a prime element in

$R_0[s, s^{-1}, f_0^{-1}] = Z$. Furthermore, p does not divide $g = 1 - sf_0$ in $R_0[s]$, and hence not in Z. Therefore, Equation (1) implies that p divides b_{-1}. After cancelling p, the same equation shows that g divides a_{-1}. Hence the ideal (p, g) of Z contains the ideal (a_{-1}, b_{-1}). Similarly, Equation (2) implies that p divides d_0 in Z, then g divides c_0 and so (p, g) also contains the ideal (c_0, d_0) of Z. However, (3) implies that $(a_{-1}, b_{-1}, c_0, d_0) = (1)$. Hence also $(p, g) = (1)$ in Z. This means that there exist polynomials $x, y \in R_0[s]$ and an integer $n \geq 0$ such that $px + gy = (sf_0)^n$. Passing to the respective images modulo p and denoting them by overlining, we obtain $\overline{gy} = (\overline{s}\overline{f}_0)^n$ in $(R_0/pR_0)[s]$. However, since $g = 1 - sf_0$, we also have $\overline{gz} = \overline{1} - (\overline{s}\overline{f}_0)^n$ with $z = 1 + sf_0 + \ldots + (sf_0)^{n-1} \in R_0[s]$. Thus $\overline{g}(\overline{y} + \overline{z}) = \overline{1}$, showing that $\overline{g}$ is a unit in $(R_0/pR_0)[s]$. So $\overline{g} = \overline{1} - \overline{s}\overline{f}_0 \in (R_0/pR_0)^*$, implying $\overline{s}\overline{f}_0 = \overline{0}$ in $(R_0/pR_0)[s]$. Therefore p divides sf_0 and hence f_0 in $R_0[s]$. However, this contradicts our assumption on p. Hence $a \in HB$ is impossible and our claim is proved.

Now we set $b := \begin{pmatrix} 1 & p^2t^{-1} \\ 0 & 1 \end{pmatrix}$, which is certainly an element of $B = SL_2\begin{pmatrix} \mathcal{O} & t^{-1}\mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix}$ but not of $U = SL_2\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{pmatrix}$. Finally we check that $aba^{-1} = \begin{pmatrix} 1 - pgt^{-1} & p^2t^{-1} \\ -g^2t^{-1} & 1 + pgt^{-1} \end{pmatrix}$, which is certainly an element of $H = SL_2(S)$. So by Lemma 2.6, $aba^{-1} \notin H_0$, and hence in particular (since $E_2(S) \leq H_0$ as remarked above) $aba^{-1} \notin E_2(S)$. $\square$

Remark 4.6 The proof of Proposition 4.5 in fact yields many more matrices which are not contained in $E_2(S)$. For instance, let $b' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be any element of $SL_2(R_0[s])$ satisfying $\alpha \equiv \delta \bmod p$, $\beta \equiv 0 \bmod p^2$ and $\beta \neq 0$. Then $b = \begin{pmatrix} \alpha & \beta t^{-1} \\ \gamma t & \delta \end{pmatrix}$ is an element of B, but not of U, and again $aba^{-1} \in SL_2(S)$ (with a as in the proof of Proposition 4.5) and $aba^{-1} \notin E_2(S)$.

One should also note that all these matrices aba^{-1} (including the one given in Proposition 4.5) are in fact elements of $SL_2(R_0[s, t^{-1}])$; their entries only involve s and t^{-1} but not s^{-1} and t.

We are now in a position to prove the main result of this section, stated as Theorem 1.2 in the Introduction.

Theorem 4.7 *Let R_0 be a finitely generated integral domain with infinitely many non–associate prime elements, $R = R_0[s, t]$ and F the field of fractions of R. Then $SL_2(R)$ is not contained in $E_2(S)$ for any finitely generated R–subalgebra S of F. Equivalently, Γ is not a finitely generated group whenever $SL_2(R) \leq \Gamma \leq SL_2(F)$.*

Proof. We first prove the claim for $R' := R_0[s, t^{-1}]$ instead of R. Any finitely generated R'–algebra $S' \subseteq F$ is contained in one of the form $S'' = R_0[s, t, t^{-1}, f^{-1}]$ with a nonzero polynomial $f \in R_0[s, t]$. (Write the generators of S' as fractions $g_1/f_1, \ldots, g_n/f_n$ with polynomials $g_i, f_i \in R_0[s, t]$ and define f to be the product $f = f_1 \ldots f_n$.) We may additionally assume that t does not divide f since $t, t^{-1} \in S''$. Now define $f_0 \in R_0[s] \setminus \{0\}$ as in Proposition 4.5 and set again $S = R_0[s, t, s^{-1}, t^{-1}, f_0^{-1}, f^{-1}]$. Since $f_0 \neq 0$, it must have at least one nonzero coefficient c in R_0. Because c is only divisible by finitely many non–associate prime elements of R_0 (R_0 is noetherian), our assumption on R_0 guarantees the existence of a prime element p of R_0 not dividing c and hence also not dividing f_0 in $R_0[s]$. Now Proposition 4.5 provides us with an element of $SL_2(R')$ (see the last paragraph of the previous remark) which is not contained in $E_2(S)$ and hence also not in $E_2(S')$.

So $SL_2(R')$ is not contained in $E_2(S')$ for any finitely generated R'–algebra $S' \subseteq F$. However, the roles of t and t^{-1} are of course symmetric in this situation (more formally: consider the automorphism of F interchanging t and t^{-1} and fixing $R_0[s]$ pointwise). Hence the analogous statement for $SL_2(R)$ is also true. Finally, the equivalence of this with the last statement of the theorem was already established in Lemma 4.3. □

Corollary 4.8 *Let R be one of the following polynomial rings*

(1) $R = \mathbb{Z}[t_1, \ldots, t_m]$ *with* $m \geq 2$

(2) $R = \mathbb{F}_q[t_1, \ldots, t_m]$ *with* $m \geq 3$

and F its field of fractions. Then any group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F)$ is not finitely generated.

More generally, if $R = R'_0[t_1, \ldots, t_m]$ for an arbitrary finitely generated integral domain R'_0 and $m \geq 3$ or if $R = R_0[t_1, t_2]$ for a Hasse domain (= S–arithmetic ring) R_0, then the same result holds for R. (Recall that any Hasse domain has infinitely many non–associate primes by a well-known theorem from number theory.)

Remark 4.9 It is interesting to note that for any Hasse domain R_0 and any $m > 0$, $SL_n(R) = E_n(R)$ is finitely generated for $R = R_0[t_1, \ldots, t_m]$ and *all* $n \geq 3$. This was shown by Suslin in [11].

I do not know how restrictive the assumption in Theorem 4.7 concerning the infinitely many primes really is. It might well be that *any* finitely generated infinite integral domain

has infinitely many non–associate prime elements. However, I have not yet found a reference yielding this statement in this generality.

We conclude this section by strengthening the statement $SL_2(S) \neq E_2(S)$ for finitley generated R–subalgebras S of F similarly as Bachmuth and Mochizuki did for Laurent polynomial rings (see [1, Theorem 1]).

Corollary 4.10 *Let R and F be as in Theorem 4.7, and let S be a finitely generated R–subalgebra of F. Then any set of generators of $SL_2(S)$ must contain infinitely many elements outside $E_2(S)$.*

Proof. If there were a finite subset $L \subset SL_2(S)$ such that $SL_2(S) = \langle E_2(S) \cup L \rangle$, then there were also a finite subset $M \subset F$ such that $SL_2(S) \leq E_2(S[M])$ (see the proof of Lemma 4.2(i)), implying $SL_2(R) \leq E_2(S[M])$. Since also $S[M]$ is a finitely generated R–algebra, the latter inclusion is impossible by Theorem 4.7. □

5 Some problems and conjectures

One does not necessarily need polynomial rings in at least two variables in order to get similar results (with, however, more technical proofs) as stated in Proposition 4.5 and Theorem 4.7. They all support the following

Conjecture 5.1 If R is a finitely generated integral domain of Krull dimension at least 3 with field of fractions F, then a group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F)$ is never finitely generated.

The following question is a natural generalization of Rapinchuk's original problem in the case $\mathcal{G} = SL_2$:

Question 5.2 Is it possible in the situation of Conjecture 5.1 that there exists a finitely generated group Γ with $SL_2(R) \leq \Gamma \leq SL_2(F')$ if we admit *any* field F' which contains R?

Whereas the rings of algebraic number theory are very well analyzed (see Remark 4.4), the situation is pretty unclear for (finitely generated) domains of *Krull dimension 2*. I think that one only has a chance to attack Question 4.1 for this class of rings if one has settled

the following two very concrete (but hard!) problems which were already mentioned in Remark 3.11. I state the first of these two problems as a conjecture since some numerical evidence (which unfortunately did not lead to a systematic proof) makes me believe that it is a true statement.

Conjecture 5.3 $SL_2(\mathbb{Z}[t, t^{-1}]) = E_2(\mathbb{Z}[t, t^{-1}])$.

Question 5.4 Is $SL_2(\mathbb{F}_q[t_1, t_1^{-1}; t_2, t_2^{-1}])$ generated by elementary matrices or at least finitely generated?

Rapinchuk's problem for $\mathcal{G} = SL_n$ with $n \geq 3$ is also challenging but of a completely different nature. At least for "sufficiently large" n it purely becomes a question of algebraic K-theory, namely the following.

Question 5.5 If R is a finitely generated integral domain, is there always an element $0 \neq f \in R$ such that $K_1(R[1/f])$ is a finitely generated (abelian) group?

Remark 5.6 If Question 5.5 has a positive answer for a given finitely generated integral domain R with Krull dimension d and if $n \geq d+2$, then $SL_n(R[1/f])$ is a *finitely generated* group containing $SL_n(R)$.

A positive answer to Question 5.5 is known for many rings R. However, it seems to be a hard problem in general. A solution would be immediately provided by a positive answer to the following more general question asked by Bass thirty years ago (see [2], problem at the end of the introduction): Is $K_1(S)$ finitely generated for any *regular* finitely generated commutative ring S? (It is well known from commutative algebra that for any finitely generated integral domain R, there exists $0 \neq f \in R$ such that $R[1/f]$ is regular.)

We close this section (and this paper) by returning, in a very special case, to anisotropic groups, which originally motivated Rapinchuk's Question 1.1.

Conjecture 5.7 Let R and F be as in Conjecture 5.1, let D be a quaternion algebra over F, and consider $\mathcal{G} = SL_{1,D}$, the (algebraic) group of elements of reduced norm 1. Then there does not exist a finitely generated group Γ with $\mathcal{G}(R) \leq \Gamma \leq \mathcal{G}(F)$.

This conjecture is motivated by Theorem 4.7, Proposition 4.5 and Remark 4.6. I have no idea yet what results are to be expected for $SL_{1,D}$ in case the degree of D is > 2.

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